

# S-duality Improved Superstring Perturbation Theory

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## Abstract

Strong - weak coupling duality in string theory allows us to compute physical quantities both at the weak coupling end and at the strong coupling end. Furthermore perturbative string theory can be used to compute corrections to the leading order formula at both ends. We explore the possibility of constructing a smooth interpolating formula that agrees with the perturbation expansion at both ends and leads to a fairly accurate determination of the quantity in consideration over the entire range of the coupling constant. We apply this to study the mass of the stable non-BPS state in SO(32) heterotic / type I string theory with encouraging results. In particular our result suggests that after taking into account one loop corrections to the mass in the heterotic and type I string theory, the interpolating function determines the mass within 10% accuracy over the entire range of coupling constant.

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## 1 Introduction

At present we do not have a fully non-perturbative definition of string theory except in some special backgrounds involving AdS spaces. As a result we can only compute perturbative corrections around a given background (see [1] for up to date results on superstring perturbation theory) which is expected to break down at finite or strong coupling. S-duality provides a way out at strong coupling by mapping the problem to a weak coupling problem in a dual string theory. However there are no known techniques for systematic computation at finite coupling.

Certain supersymmetric quantities can be computed for all values of the coupling since once they are computed at weak coupling they remain valid at all couplings. This includes spectrum of BPS states, certain terms in the low energy effective action (*e.g.* the prepotential in type II string theory compactified on a Calabi-Yau 3-fold) etc. Recently remarkable progress has been made towards determining certain other class of terms in the low energy effective action whose form is not protected against quantum corrections, but are sufficiently constrained by supersymmetry so that by knowing the perturbative answer at various ends we can completely fix these terms (see *e.g.* [2] and references therein). However this still leaves open the question of how to determine the wide class of other observables whose form is not in any way restricted by supersymmetry. Most of the interesting observables like the S-matrix and masses of non-BPS states fall in this category.

In this paper we explore the possibility that knowing the behaviour at strong and weak coupling and matching the results from two ends, we may be able to get fairly accurate results for physical quantities even at finite coupling. The idea is as follows. Let us denote by  $F_m^W(g)$  and  $F_n^S(g)$  respectively the contributions to a given physical quantity up to  $m$ -th order at the weak coupling end ( $g \rightarrow 0$ ) and  $n$ -th order at the strong coupling end ( $g \rightarrow \infty$ ). We then try to look for a smooth interpolating function  $F_{m,n}(g)$  whose Taylor series expansions at the weak and the strong coupling ends match those of the functions  $F_m^W(g)$  and  $F_n^S(g)$  to appropriate order. Under favourable circumstances the function  $F_{m,n}(g)$  may come reasonably close to the actual function  $F(g)$  for sufficiently large  $m, n$ . Since the perturbation expansion in string theory is an asymptotic expansion, we do not expect that we can approach arbitrarily close to the exact result; but the question is whether we can reach fairly close to the exact result following this procedure.<sup>1</sup>

We shall apply this procedure to study the mass of the lightest  $\text{SO}(32)$  spinor state in  $\text{SO}(32)$  heterotic or equivalently type I string theory. Due to charge conservation this state is guaranteed to be stable even though it breaks all supersymmetry. In  $\text{SO}(32)$  heterotic string theory this is a perturbative string state [4–6] whereas in type I string theory this is described by a stable non-BPS D0-brane [7, 8] (see also [9–12]). Thus it is meaningful to look for a function  $F(g)$  that will give the mass of this state as a function of the string coupling constant  $g$ .

The rest of the paper is organised as follows. In §2 we fix our conventions, describe our strategy for finding the interpolating function and also explicitly find the interpolating function at the leading order. In §3 we compute first subleading correction to the mass of the stable non-BPS state in type I string theory. In §4 we find the first subleading correction to the mass of stable non-BPS state in  $\text{SO}(32)$  heterotic string theory. In §5 we find the interpolating function taking into account the subleading corrections at the two ends and compare the result with the leading order interpolating function, as well as the strong and weak coupling expansions. We find close matching of all these functions within about 10%, indicating that already at this order the interpolating function may be within 10% of the exact result for all values of the coupling. In appendix A we test the efficiency of our interpolation algorithm by applying it on several test functions. Appendix B contains some technical details of the analysis carried out in §4.

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<sup>1</sup>In a different context but similar spirit, ref. [3] attempted to find an approximate formula for the negative mode eigenvalue of the Schwarzschild black hole as a function of dimension  $D$  using the known behaviour at large and small  $D - 3$ .

## 2 Conventions and Strategy

We begin by fixing the various normalization conventions we shall be using in our analysis. We denote by  $G_{H\mu\nu}$  and  $G_{I\mu\nu}$  the heterotic and type I metric, defined so that the fundamental strings in the respective string theories have tension  $1/2\pi$  in these metrics. We shall choose the dilatons  $\phi_H$  and  $\phi_I$  in the two theories so that the part of the action involving the metric, dilaton and the  $\text{SO}(32)$  gauge fields take the form:

$$S_H = \int d^{10}x e^{-2\phi_H} \sqrt{\det G_H} \left[ \frac{1}{2} R_H + 2G_H^{\mu\nu} \partial_\mu \phi_H \partial_\nu \phi_H - \frac{1}{16} G_H^{\mu\nu} G_H^{\rho\sigma} \text{Tr}_V(F_{\mu\rho} F_{\nu\sigma}) \right] \quad (2.1)$$

for the heterotic string theory and

$$S_I = \int d^{10}x \sqrt{\det G_I} \left[ e^{-2\phi_I} \left\{ \frac{1}{2} R_I + 2G_I^{\mu\nu} \partial_\mu \phi_I \partial_\nu \phi_I \right\} - C e^{-\phi_I} G_I^{\mu\nu} G_I^{\rho\sigma} \text{Tr}_V(F_{\mu\rho} F_{\nu\sigma}) \right] \quad (2.2)$$

for the dual type I string theory. Here  $\text{Tr}_V$  denotes trace in the vector representation of  $\text{SO}(32)$ ,

$$C = 2^{-13/2} \pi^{-7/2} \quad (2.3)$$

and  $R_H$  and  $R_I$  denotes the scalar curvatures computed from the heterotic and type I metrics respectively. Note that the overall normalization of the terms involving the metric and the dilaton can be changed by shifting  $\phi_H$  and  $\phi_I$ , but once these terms have been fixed the normalization of the gauge field kinetic term is no longer arbitrary. For the heterotic string theory this normalization was determined in [6] while for type I string theory this can be found *e.g.* in [13].<sup>2</sup>

We now introduce the Einstein metric  $g_{\mu\nu}$  via the field redefinitions

$$G_{H\mu\nu} = e^{\phi_H/2} g_{\mu\nu}, \quad G_{I\mu\nu} = e^{\phi_I/2} g_{\mu\nu}. \quad (2.4)$$

In terms of the metric  $g_{\mu\nu}$  the action takes the form

$$S_H = \int d^{10}x \sqrt{\det g} \left[ \frac{1}{2} R - \frac{1}{4} g^{\mu\nu} \partial_\mu \phi_H \partial_\nu \phi_H - \frac{1}{16} e^{-\phi_H/2} g^{\mu\nu} g^{\rho\sigma} \text{Tr}_V(F_{\mu\rho} F_{\nu\sigma}) \right] \quad (2.5)$$

and

$$S_I = \int d^{10}x \sqrt{\det g} \left[ \frac{1}{2} R - \frac{1}{4} g^{\mu\nu} \partial_\mu \phi_I \partial_\nu \phi_I - C e^{\phi_I/2} g^{\mu\nu} g^{\rho\sigma} \text{Tr}_V(F_{\mu\rho} F_{\nu\sigma}) \right]. \quad (2.6)$$

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<sup>2</sup>In [13] the action was written as a 8+1 dimensional integral in the T-dual type I' description. Here we have expressed the action as a 9+1 dimensional integral in type I description.

Comparing (2.5) and (2.6) we see that the two actions agree if we make the identification

$$e^{(\phi_H + \phi_I)/2} = (16C)^{-1}. \quad (2.7)$$

We shall define the heterotic coupling  $g_H$  and the type I coupling  $g_I$  via the relations

$$g_H \equiv e^{\langle \phi_H \rangle}, \quad g_I = e^{\langle \phi_I \rangle}. \quad (2.8)$$

Eqs.(2.7), (2.3) then give

$$g_H g_I = 2^{-8} C^{-2} = 2^5 \pi^7. \quad (2.9)$$

To test the duality between the heterotic and type I string theory, we can compare the fundamental heterotic string tension with the type I D-string tension. Heterotic string tension in Einstein frame is given by

$$T_H = \frac{1}{2\pi} (g_H)^{1/2}. \quad (2.10)$$

On the other hand the type I D-string tension in Einstein frame is given by [14, 15]

$$T_D = \frac{1}{2} (2\pi)^{5/2} (g_I)^{-1/2}. \quad (2.11)$$

Note that there is an extra factor of  $1/\sqrt{2}$  compared to that of the tension of the type IIB D-string to take into account the effect of orientifold projection. Using (2.9) we see that  $T_D = T_H$ , in agreement with the heterotic - type I duality [16] that identifies the fundamental heterotic string with the type I D-string.

From now on all the masses will be given in the Einstein metric unless mentioned otherwise. In the convention described above, the mass of the lightest  $\text{SO}(32)$  spinor state in the heterotic string theory is given by

$$2 (g_H)^{1/4} (1 + \mathcal{O}(g_H^2)). \quad (2.12)$$

On the other hand the mass of the stable non-BPS D0-brane in type I string theory, transforming in the spinor representation of  $\text{SO}(32)$ , can be computed as follows. To leading order in  $g_I$ , the D0-brane is connected by marginal deformation to a D-string anti-D-string pair wrapped on a compact circle of radius  $1/\sqrt{2}$  measured in the type I metric [7]. Thus its mass, measured in the type I metric, will be  $2\pi\sqrt{2}$  times the tension of a D-string. Converting this to the Einstein metric the mass of the D0-brane is given by

$$2\sqrt{2} \pi T_D (g_I)^{-1/4} (1 + \mathcal{O}(g_I)) = 2^3 \pi^{7/2} (g_I)^{-3/4} (1 + \mathcal{O}(g_I)) = 2^{-3/4} \pi^{-7/4} (g_H)^{3/4} (1 + \mathcal{O}(g_H^{-1})). \quad (2.13)$$

Comparing (2.13) and (2.12) we see that the leading order heterotic and type I results meet at  $g_H = (2\pi)^{7/2}$ . In view of this we introduce a rescaled coupling parameter  $g$  via

$$g_H = 2^{7/2}\pi^{7/2}g, \quad g_I = 2^{3/2}\pi^{7/2}g^{-1}, \quad (2.14)$$

so that the two formulæ meet at  $g = 1$ . Furthermore since at the meeting point the mass is given by  $2^{15/8}\pi^{7/8}$ , we defined a renormalized mass function  $F(g)$  via the relation

$$M(g) = 2^{15/8}\pi^{7/8}F(g). \quad (2.15)$$

In terms of  $g$  the leading order weak and the strong coupling formulæ (2.12) and (2.13) for the renormalized mass function  $F(g)$  can be expressed as

$$F_0^W(g) = g^{1/4}, \quad F_0^S(g) = g^{3/4}. \quad (2.16)$$

Our goal will be to explore to what extent we can determine the full function  $F(g)$  by finding an interpolating function that matches onto the above functions (and perturbative corrections to them) at the two ends.

Let us denote by  $F_m^W(g)$  and  $F_n^S(g)$  the formulæ for  $F(g)$  in the weak and strong coupling limits to  $m$ -th and  $n$ -th order in expansion in powers of  $g$  and  $g^{-1}$  respectively. In that case we shall choose our interpolating function as<sup>3</sup>

$$F_{m,n}(g) = g^{1/4} \left[ 1 + a_1g + \cdots + a_mg^m + b_ng^{m+1} + b_{n-1}g^{m+2} + \cdots + b_1g^{m+n} + g^{m+n+1} \right]^{1/\{2(m+n+1)\}}. \quad (2.17)$$

By construction this formula reduces to the correct forms given in (2.12) and (2.13) in the weak and strong coupling limits. The coefficients  $a_1, \dots, a_m$  are determined by demanding that the expansion of  $F_{m,n}$  in powers of  $g$  matches the weak coupling perturbation expansion to  $m$ -th order, while the coefficients  $b_1, \dots, b_n$  are determined by demanding that the expansion of  $F_{m,n}$  in powers of  $1/g$  matches the strong coupling perturbation expansion to  $n$ -th order.<sup>4</sup> Since in the heterotic string theory the expansion is actually in powers of  $g^2$ , this will imply vanishing of the  $a_m$ 's for odd  $m$ .

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<sup>3</sup>Clearly many other interpolations are possible. In particular we could use (fractional) power of a rational function for this purpose. In each case we need to determine the efficiency of the interpolating algorithm by studying its convergence properties. As we shall see, for the problem of studying the mass of lightest SO(32) spinor states in heterotic string theory, (2.17) seems to give reasonable results.

<sup>4</sup>Note that the  $a_k$ 's and  $b_k$ 's which appear in the analysis of each  $F_{m,n}$  are different and have to be determined afresh every time.

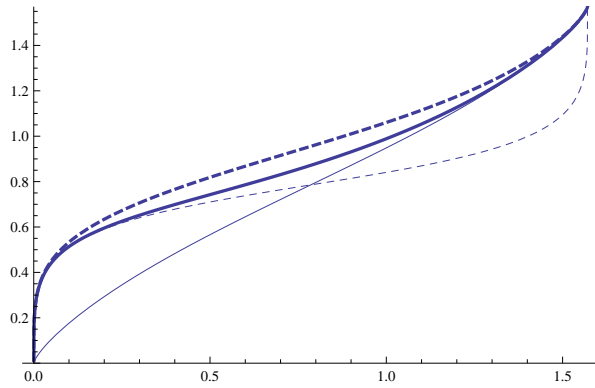


Figure 1: Graph of  $\tan^{-1} F(g)$  vs.  $\tan^{-1} g$  for  $F = F_0^S$  (thin solid curve),  $F_0^W$  (thin dashed curve) and the interpolating functions  $F_{0,0}$  (thick dashed curve) and  $F_{1,0}$  (thick solid curve). This should be compared with Fig. 2 which describes similar curves after taking into account the first subleading corrections at both ends.

Note that this procedure is not foolproof since the term inside the square bracket in (2.17) could become negative and hence the right hand side of (2.17) will cease to give a real function. This will signal breakdown of this procedure. However for sufficiently smooth functions we could expect such negative  $a_k$  and/or  $b_k$  coefficients to be small even if they are present. In this case the term inside the square bracket in (2.17) will remain positive for all positive  $g$  and the procedure should continue to work. Nevertheless this approach will clearly be insensitive to terms in  $F(g)$  whose Taylor series expansion vanishes at both ends, *e.g.*  $e^{-Ag-B/g}$  times any polynomial in  $g, g^{-1}$  for positive constants  $A, B$ . More generally since the perturbation expansion in string theory is expected to represent asymptotic series at both ends, we do not expect to get arbitrarily close to the exact result by going to arbitrarily high order. Typically for any given value of  $g$  the best result will be reached at some particular order in the perturbation theory. The hope is that this approach may take us sufficiently close to the exact formula over the entire range of  $g$ . In appendix A we have tested this procedure on several test functions.

For example for the leading order functions (2.16) we have the interpolating function

$$F_{0,0}(g) = g^{1/4}(1+g)^{1/2}. \quad (2.18)$$

Also knowing that the weak coupling expansion begins at order  $g^2$  so that  $a_1 = 0$ , we get

$$F_{1,0}(g) = g^{1/4}(1+g^2)^{1/4}. \quad (2.19)$$

In Fig. 1 we have plotted the functions  $F_0^S(g)$ ,  $F_0^W(g) = F_1^W(g)$  as well as  $F_{1,0}(g)$  and  $F_{0,0}(g)$ .

### 3 Strong coupling expansion

We shall now compute  $F_1^S(g)$  by determining the first order correction  $\Delta M$  to the mass formula from the strong coupling end, i.e. in type I string theory. This is given by open string one loop correction to the energy of the non-BPS D0-brane of type I string theory, and can be expressed as [14]<sup>5</sup>

$$-\Delta M = \frac{1}{2} g_I^{1/4} (8\pi^2)^{-1/2} \int_0^\infty s^{-3/2} ds [Z_{NS;D0D0} - Z_{R;D0D0} + Z_{NS;D0D9} - Z_{R;D0D9}] \quad (3.1)$$

where  $Z_{NS;D0D0}$ ,  $Z_{R;D0D0}$ ,  $Z_{NS;D0D9}$ ,  $Z_{R;D0D9}$  denote respectively the contributions from the NS and R sector open strings with both ends on the D0-brane and NS and R sector open strings with one end on the D0-brane and the other end on the D9-brane. Explicit calculation gives [17–19]

$$\begin{aligned} Z_{NS;D0D0} &= \frac{1}{2} \frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} + 2^{5/2} (1-i) \frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{5/2} (1+i) \frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})}, \\ Z_{R;D0D0} &= \frac{1}{2} \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8}, \\ Z_{NS;D0D9} &= 16\sqrt{2} \frac{f_1(\tilde{q}) f_2(\tilde{q})^9}{f_4(\tilde{q})^9 f_3(\tilde{q})}, \\ Z_{R;D0D9} &= 16\sqrt{2} \frac{f_1(\tilde{q}) f_3(\tilde{q})^9}{f_4(\tilde{q})^9 f_2(\tilde{q})}, \end{aligned} \quad (3.2)$$

where

$$\tilde{q} \equiv e^{-\pi s}, \quad (3.3)$$

$$\begin{aligned} f_1(q) &\equiv q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) = \eta(2\tau), \quad q \equiv e^{2\pi i \tau} \\ f_2(q) &\equiv \sqrt{2} q^{1/12} \prod_{n=1}^{\infty} (1 + q^{2n}) = \sqrt{2} \eta(4\tau) / \eta(2\tau), \\ f_3(q) &\equiv q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \eta(2\tau)^2 / (\eta(4\tau) \eta(\tau)), \end{aligned}$$

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<sup>5</sup>The variable  $s$  is related to the variable  $t$  appearing in eq.(9) of [14] as  $t = 4\pi s$ . There is a factor of  $1/2$  in our expression compared to that in [14] since we are considering the self energy of the D0-brane, and hence we do not get a factor of 2 in the spectrum by exchanging the two ends of the string. The explicit form of the open string partition function is of course different since we have a non-BPS D0-brane in type I string theory while [14] was considering BPS D-p-branes in type II string theory.



$$f_4(q) \equiv q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}) = \eta(\tau)/\eta(2\tau). \quad (3.4)$$

Individual terms in (3.1) are both infrared (IR) and ultraviolet (UV) divergent; so we need to be careful about using the right prescription for the IR and UV regularization. Since  $2\pi s$  denotes the proper length of the open string propagator flowing in the loop in the Schwinger representation, the IR regularization is done by putting a uniform upper cut-off  $\Lambda$  on  $s$  in all the integrals. Regulating the UV divergence is more subtle. As we shall describe shortly, instead of putting a uniform lower cut-off on the integrals, the correct prescription is to use a lower cut-off  $\epsilon$  on all the integrals involving  $Z_{R;D_0D_0}$ ,  $Z_{NS,D_0D_9}$ ,  $Z_{R;D_0D_9}$  and the first term in  $Z_{NS;D_0D_0}$ , but use a cut-off  $\epsilon/4$  on the integrals involving the second and third terms of  $Z_{NS;D_0D_0}$ . Thus the integral appears as

$$-\Delta M = -\tilde{K}_s (g_I)^{1/4} \quad (3.5)$$

where

$$\begin{aligned} \tilde{K}_s \equiv & -\frac{1}{2} (8\pi^2)^{-1/2} \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left[ \int_{\epsilon}^{\Lambda} s^{-3/2} ds \left\{ \frac{1}{2} \frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} - \frac{1}{2} \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8} \right. \right. \\ & + 16\sqrt{2} \frac{f_1(\tilde{q})f_2(\tilde{q})^9}{f_4(\tilde{q})^9 f_3(\tilde{q})} - 16\sqrt{2} \frac{f_1(\tilde{q})f_3(\tilde{q})^9}{f_4(\tilde{q})^9 f_2(\tilde{q})} \Big\} \\ & \left. + \int_{\epsilon/4}^{\Lambda} s^{-3/2} ds \left\{ 2^{5/2}(1-i) \frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{5/2}(1+i) \frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})} \right\} \right]. \quad (3.6) \end{aligned}$$

Using known modular transformation laws of the  $f_i$ 's we can also express (3.6) in the 'closed string channel':

$$\tilde{K}_s = - \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi} (8\pi^2)^{-1/2} \left[ \int_{\pi/\Lambda}^{\pi/\epsilon} dt (C_{00} + C_{09} + C_{09}^*) + \int_{\pi/4\Lambda}^{\pi/\epsilon} dt (\mathcal{M} + \mathcal{M}^*) \right], \quad (3.7)$$

where

$$\begin{aligned} C_{00} &= \left( \frac{\pi}{t} \right)^{9/2} \left[ \frac{f_3(q)^8}{f_1(q)^8} - \frac{f_4(q)^8}{f_1(q)^8} \right], \\ \mathcal{M} &= 2^{9/2} \left[ \frac{f_3(iq)^9 f_1(iq)}{f_2(iq)^9 f_4(iq)} - \frac{f_4(iq)^9 f_1(iq)}{f_2(iq)^9 f_3(iq)} \right], \\ C_{09} &= 2^{9/2} \left[ \frac{f_4(q)^9 f_1(q)}{f_2(q)^9 f_3(q)} - \frac{f_3(q)^9 f_1(q)}{f_2(q)^9 f_4(q)} \right], \\ q &\equiv e^{-t}. \end{aligned} \quad (3.8)$$

In going from (3.6) to (3.7) we have made a change of variables  $t = \pi/s$  in the first integral and  $t = \pi/4s$  in the second integral, explaining the limits of integration in (3.7). Physically  $C_{00}$  denotes the cylinder amplitude with both boundaries lying on the D0-brane, given by the inner product between the boundary states of the D0-brane,  $\mathcal{M}$  denotes the Mobius strip amplitude with the boundary lying on the D0-brane, given by the inner product between the boundary states of the D0-brane and the crosscap, and  $C_{09}$  is the cylinder amplitude with one boundary on the D0-brane and the other boundary on the D9-brane, given by the inner product between the boundary states of the D0-brane and the D9-brane. In (3.7) the parameter  $t$  has the interpretation of the proper length of the closed string propagator in the Schwinger representation, and hence the upper cut-off on  $t$  should be a uniform number for all the integrals. This is indeed the case in (3.7) since all the upper cut-offs are  $\pi/\epsilon$ , but this required choosing the lower cut-off on the  $s$  integral in precisely the way we have chosen in (3.6).

Expressing (3.5) in terms of  $g$  defined in (2.14), carrying out the rescaling given in (2.15), and combining this with the leading order strong coupling result given in (2.16) we see that the strong coupling result for  $F(g)$  corrected to first order in  $1/g$  is given by

$$F_1^S(g) = g^{3/4} (1 + K_s g^{-1}), \quad K_s \equiv 2^{-3/2} \tilde{K}_s. \quad (3.9)$$

Numerical evaluation of the integrals appearing in (3.6) gives

$$K_s \simeq .351. \quad (3.10)$$

A graph of  $\tan^{-1} F_1^S(g)$  vs.  $\tan^{-1} g$  can be found in Fig. 2, but we shall postpone the analysis of this function till §5 by which time we shall also compute the first subleading corrections from the weak coupling end.

## 4 Weak coupling expansion

Next we shall compute the first order correction to  $M$  in the weakly coupled heterotic string theory leading to an expression for  $F_2^W(g) = F_3^W(g)$ . The left-handed part of the state is created by the spin field of  $SO(32)$  acting on the vacuum. The right handed part can be chosen in different ways. In the light cone gauge Neveu-Schwarz-Ramond (NSR) formalism the NS sector states can be taken to be of the form [5]

$$\psi_{-3/2}^i |0\rangle, \quad \psi_{-1/2}^i \psi_{-1/2}^j \psi_{-1/2}^k |0\rangle, \quad \psi_{-1/2}^i \alpha_{-1}^j |0\rangle \quad (4.1)$$

where  $\psi_{-n}^i$  and  $\alpha_{-n}^i$  are transverse fermionic and bosonic oscillators respectively with  $1 \leq i \leq 8$ . This gives a total of  $8+56+64=128$  states. Together with the 128 fermionic states arising from the R-sector, these form a single long supermultiplet transforming in the spinor representation of the gauge group  $SO(32)$ . Thus all the states suffer the same mass renormalization and we can focus on any one of them for computing the correction to  $M$ . We shall use covariant NSR formulation [20] for our computation and choose the unintegrated vertex operator in the  $-1$  picture to be

$$i c \bar{c} e^{-\varphi} \bar{S}_\alpha(\bar{z}) \psi^1(z) \psi^2(z) \psi^3(z) e^{ik_0 X^0} \quad (4.2)$$

where  $\varphi$  is the bosonized ghost of the  $\beta$ - $\gamma$  system [20],  $b, \bar{b}, c, \bar{c}$  are the diffeomorphism ghosts,  $\bar{S}_\alpha$  are the spin fields of  $SO(32)$  and  $k^0$  is the energy of the particle in the rest frame given by  $M$ . The unintegrated vertex operator in the zero picture is then given by

$$\begin{aligned} c\bar{c}V_0(k, z) = i c \bar{c} e^{ik_0 X^0} \bar{S}_\alpha(\bar{z}) \Big\{ \psi^1(z) \psi^2(z) \partial X^3(z) + \psi^2(z) \psi^3(z) \partial X^1(z) \\ + \psi^3(z) \psi^1(z) \partial X^2(z) + ik_0 \psi^1(z) \psi^2(z) \psi^3(z) \psi^0(z) \Big\}, \end{aligned} \quad (4.3)$$

plus terms of order  $e^\varphi$  whose correlation functions vanish by  $\varphi$ -charge conservation. For computing the torus amplitude we need to convert one of the vertex operators to integrated vertex operator  $\tilde{V}_0$  by removing the  $c\bar{c}$  factor. If  $\delta M^2$  denotes the one loop correction to  $M^2$ , then up to an overall multiplicative factor  $\delta M^2$  is given by

$$\delta M^2 \sim (g_H)^2 \int d^2\tau \int d^2z \langle V_0(-k, 0) \tilde{V}_0(k, z) \rangle_{matter} Z_{ghost}, \quad (4.4)$$

where  $\tau$  is the modular parameter of the torus, integrated over the fundamental domain,  $\langle \rangle_{matter}$  denotes correlation function in the matter sector on the torus multiplied by the matter partition function and  $Z_{ghost}$  denotes the ghost partition function after removal of ghost zero modes.

To evaluate this contribution, let us first focus on the contribution from the holomorphic fermions  $\psi^\mu(z)$ . The possibly non-vanishing correlation functions are of two types:  $\langle \psi^1(0) \psi^2(0) \psi^1(z) \psi^2(z) \rangle$  and its permutations and  $\langle \psi^1(0) \psi^2(0) \psi^3(0) \psi^0(0) \psi^1(z) \psi^2(z) \psi^3(z) \psi^0(z) \rangle$ . To evaluate these we first perform a double Wick rotation on  $\psi^0$  to make it  $\psi^4(0)$  corresponding to some Euclidean direction 4, at the cost of picking up a factor of  $i$ . Next we introduce complex fermions  $\chi^k$  via  $\chi^k = (\psi^k + i \psi^{k+4})/\sqrt{2}$  so that we can express  $\psi^k$  as  $(\chi^k + \bar{\chi}^k)/\sqrt{2}$  for  $1 \leq k \leq 4$ . Then we have

$$\langle \psi^1(0) \psi^2(0) \psi^1(z) \psi^2(z) \rangle = -\langle \psi^1(0) \psi^1(z) \psi^2(0) \psi^2(z) \rangle$$

$$\begin{aligned}
&= -\frac{1}{4} \left( \langle \chi^1(0) \bar{\chi}^1(z) \chi^2(0) \bar{\chi}^2(z) \rangle + \langle \bar{\chi}^1(0) \chi^1(z) \chi^2(0) \bar{\chi}^2(z) \rangle \right. \\
&\quad \left. + \langle \chi^1(0) \bar{\chi}^1(z) \bar{\chi}^2(0) \chi^2(z) \rangle + \langle \bar{\chi}^1(0) \chi^1(z) \bar{\chi}^2(0) \chi^2(z) \rangle \right). \tag{4.5}
\end{aligned}$$

Now for a single complex fermion like  $\chi^1(z)$  the correlation function on the torus is given by, up to a phase,

$$\langle \chi^1(0) \bar{\chi}^1(z) \rangle = (\eta(\tau))^{-1} \frac{\vartheta'_{11}(0)}{\vartheta_{11}(z)} \vartheta_\nu(z), \quad \langle \bar{\chi}^1(0) \chi^1(z) \rangle = (\eta(\tau))^{-1} \frac{\vartheta'_{11}(0)}{\vartheta_{11}(z)} \vartheta_\nu(-z), \tag{4.6}$$

where  $\nu$  denotes the spin structure on the torus taking values 00, 01, 10 and 11 and  $\vartheta_\nu$  are the Jacobi theta functions. After combining the contribution from all the holomorphic fermions and the superconformal ghosts, and taking into account the  $\tau$ -dependent normalization factors the result is, up to an overall constant factor,

$$\begin{aligned}
&\langle \psi^1(0) \psi^2(0) \psi^1(z) \psi^2(z) \rangle \\
&= -\frac{1}{4} (\eta(\tau))^{-4} \left( \frac{\vartheta'_{11}(0)}{\vartheta_{11}(z)} \right)^2 \frac{1}{2} \sum_\nu \delta_\nu \left( \vartheta_\nu(z)^2 \vartheta_\nu(0)^2 + 2\vartheta_\nu(z) \vartheta_\nu(-z) \vartheta_\nu(0)^2 + \vartheta_\nu(-z)^2 \vartheta_\nu(0)^2 \right), \tag{4.7}
\end{aligned}$$

where  $\delta_\nu = 1$  for  $\nu = 11$  and 00 and  $\delta_\nu = -1$  for  $\nu = 10$  and 01. The extra factor of  $1/2$  in (4.7) comes from the GSO projection in the right-moving (holomorphic) sector leading to the sum over spin structures  $\nu$ . Using the Riemann identity

$$\begin{aligned}
\sum_\nu \delta_\nu \vartheta_\nu(z_1) \vartheta_\nu(z_2) \vartheta_\nu(z_3) \vartheta_\nu(z_4) &= 2\vartheta_{11}((z_1 + z_2 + z_3 + z_4)/2) \vartheta_{11}((z_1 + z_2 - z_3 - z_4)/2) \\
&\quad \vartheta_{11}((z_1 - z_2 - z_3 + z_4)/2) \vartheta_{11}((z_1 - z_2 + z_3 - z_4)/2), \tag{4.8}
\end{aligned}$$

and that  $\vartheta_{11}(0) = 0$  we see that the right hand side of (4.7) vanishes.

Thus we are left with the two point function of the operators appearing in the last term inside the curly bracket in (4.3):

$$(k_0)^2 \langle e^{-ik_0 X^0(0)} e^{ik_0 X^0(z)} S^\alpha(0) S_\alpha(\bar{z}) \psi^1(0) \psi^2(0) \psi^3(0) \psi^4(0) \psi^1(z) \psi^2(z) \psi^3(z) \psi^4(z) \rangle_{matter} Z_{ghost} \tag{4.9}$$

where we have taken into account an extra  $-$  sign from the Wick rotation taking  $\psi^0$  to  $i\psi^4$ . Now following the same method as described above, the net contribution from the ten holomorphic

fermions and the superconformal ghosts to (4.9) is given by

$$\begin{aligned} \frac{1}{2^4} (\eta(\tau))^{-4} \left( \frac{\vartheta'_{11}(0)}{\vartheta_{11}(z)} \right)^4 \frac{1}{2} \sum_{\nu} \delta_{\nu} \Big( \vartheta_{\nu}(z)^4 + 4\vartheta_{\nu}(z)^3 \vartheta_{\nu}(-z) + 6\vartheta_{\nu}(z)^2 \vartheta_{\nu}(-z)^2 \\ + 4\vartheta_{\nu}(z) \vartheta_{\nu}(-z)^3 + \vartheta_{\nu}(-z)^4 \Big). \end{aligned} \quad (4.10)$$

We could manipulate this further using the Riemann identity, but for reasons that will become clear in appendix B we shall postpone this till the end.

The rest of the contribution can also be evaluated using standard method. Using the known correlator between the spin fields [21] we get the contribution from the 32 anti-holomorphic fermions to be

$$\langle \bar{S}^{\alpha}(0) \bar{S}_{\beta}(\bar{z}) \rangle = \frac{1}{2} \delta_{\alpha\beta} (\overline{\eta(\tau)})^{-16} \left( \frac{\overline{\vartheta'_{11}(0)}}{\overline{\vartheta_{11}(z)}} \right)^4 \sum_{\nu'} \overline{\vartheta_{\nu'}(z/2)^{16}}, \quad (4.11)$$

where the factor of  $1/2$  now arises from the GSO projection on the left-handed (anti-holomorphic) fermions. Finally the contribution from the ten scalars corresponding to the space-time coordinates and the diffeomorphism ghosts together give, up to a normalization,

$$(\tau_2)^{-5} \frac{1}{|\eta(\tau)|^{16}} \exp[-4\pi z_2^2/\tau_2] \left| \frac{\vartheta_{11}(z)}{\vartheta'_{11}(0)} \right|^4, \quad (4.12)$$

where we have used the on-shell condition  $(k_0)^2 = 4$ . Substituting (4.10), (4.11), (4.12) into (4.9), and using the known relation  $\vartheta'_{11}(0) = -2\pi\eta(\tau)^3$ , we can now write down the general formula for  $\delta M^2$ :

$$\begin{aligned} \delta M^2 = \frac{1}{64} \mathcal{N} (g_H)^2 (k_0)^2 \int d^2\tau \int d^2z \Big[ \left\{ (\eta(\tau))^{-4} \overline{\vartheta_{11}(z)}^{-4} \sum_{\nu'} \overline{\vartheta_{\nu'}(z/2)^{16}} \right\} \\ \left\{ \sum_{\nu} \delta_{\nu} \left( \vartheta_{\nu}(z)^4 + 4\vartheta_{\nu}(z)^3 \vartheta_{\nu}(-z) + 6\vartheta_{\nu}(z)^2 \vartheta_{\nu}(-z)^2 + 4\vartheta_{\nu}(z) \vartheta_{\nu}(-z)^3 + \vartheta_{\nu}(-z)^4 \right) \right. \\ \left. ((\eta(\tau))^8 (\vartheta_{11}(z))^{-4}) \right\} \left\{ (\eta(\tau))^{-14} (\overline{\eta(\tau)})^{-14} (\vartheta_{11}(z))^2 (\overline{\vartheta_{11}(z)})^2 \exp[-4\pi z_2^2/\tau_2] (\tau_2)^{-5} \right\} \Big] \end{aligned} \quad (4.13)$$

where  $\mathcal{N}$  is an overall normalization constant. In this expression the factor inside the first curly bracket gives the contribution from the 32 left-moving fermions, the factor inside the second curly bracket gives the contribution from the 10 right-moving fermions and the commuting superconformal ghosts, and the factor inside the third curly bracket gives the contribution

from the ten scalars and the diffeomorphism ghosts. The normalization factor  $\mathcal{N}$  has been computed in appendix B by comparing the result with the expected result in the effective field theory. The result is

$$\mathcal{N} = \frac{1}{2^{10}\pi^8}. \quad (4.14)$$

Using the fact that all the  $\vartheta_\nu$  except  $\vartheta_{11}$  are even under  $z \rightarrow -z$ , and  $\vartheta_{11}(z)$  is odd under  $z \rightarrow -z$  we can express the term in the second line of (4.13) as

$$\left(16 \sum_{\nu} \delta_{\nu} \vartheta_{\nu}(z)^4 - 16 \vartheta_{11}(z)^4\right) = -16 \vartheta_{11}(z)^4, \quad (4.15)$$

where in the last step we have again made use of the Riemann identity (4.8). Finally using the fact that  $k_0^2 = M^2$  to leading order, and using (2.14), we get

$$\delta M = M K_w g^2, \quad (4.16)$$

where

$$K_w = -\frac{1}{64\pi} \int d^2\tau \int d^2z \sum_{\nu'} \left\{ \overline{\vartheta_{\nu'}(z/2)^{16}} \right\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left( \frac{\vartheta_{11}(z)}{\overline{\vartheta_{11}(z)}} \right)^2 \exp[-4\pi z_2^2/\tau_2] (\tau_2)^{-5}. \quad (4.17)$$

It is easy to verify that the integrand is invariant under  $z \rightarrow z+1$ ,  $z \rightarrow z+\tau$  and  $\tau \rightarrow \tau+1$  and  $(z, \tau) \rightarrow (z/\tau, -1/\tau)$ . The domain of integration over  $z \equiv z_1 + iz_2$  and  $\tau \equiv \tau_1 + i\tau_2$  is the fundamental region which can be taken to be  $0 \leq z_1 < 1$ ,  $0 \leq z_2 < \tau_2$ , and the region in  $\tau$  plane bounded by the curves  $\tau_1 = \pm 1/2$  and  $|\tau| = 1$ . Numerical evaluation of the integral gives

$$K_w \simeq .23. \quad (4.18)$$

Using (2.14), (2.15), the weak coupling expansion of the mass formula  $F(g)$  up to order  $g^3$  now takes the form

$$F_2^W(g) = F_3^W(g) = g^{1/4} (1 + K_w g^2). \quad (4.19)$$

## 5 Analysis of the results and interpolating function

Let us first summarize the results of §3 and §4. We have found that the weak and strong coupling expansions of the mass function  $F(g)$  are given by

$$F_2^W(g) = g^{1/4} (1 + K_w g^2 + \mathcal{O}(g^4)), \quad F_1^S(g) = g^{3/4} (1 + K_s g^{-1} + \mathcal{O}(g^{-2})), \quad K_w \simeq .23, \quad K_s \simeq .351. \quad (5.1)$$

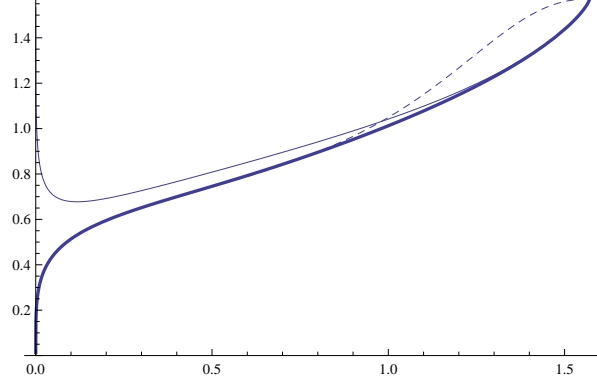


Figure 2: Graph of  $\tan^{-1} F(g)$  vs.  $\tan^{-1} g$  for  $F = F_1^S$  (thin solid curve),  $F = F_2^W (= F_3^W)$  (thin dashed curve) and the interpolating function  $F = F_{3,1}$  (the thick solid curve).

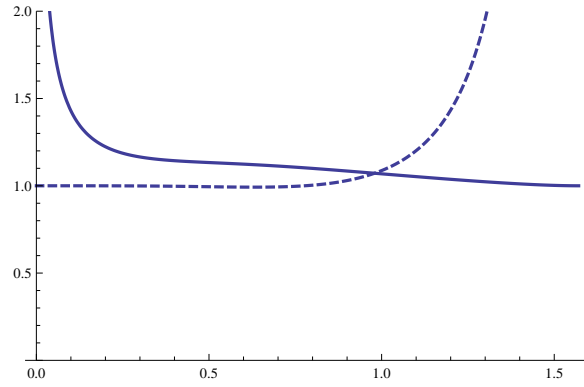


Figure 3: Graph of  $F_2^W(g)/F_{3,1}(g)$  (dashed curve) and  $F_1^S(g)/F_{3,1}(g)$  (continuous curve) vs.  $\tan^{-1} g$ .

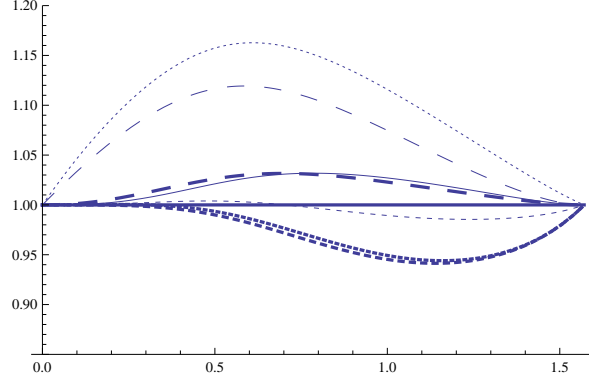


Figure 4: Graph of  $F_{m,n}(g)/F_{3,1}(g)$  vs.  $\tan^{-1} g$  for various  $(m, n)$ . The labels are as follows: thin dots for  $F_{0,0}$ , thick dots for  $F_{1,0}$ , small thin dashes for  $F_{2,0}$ , small thick dashes for  $F_{3,0}$ , large thin dashes for  $F_{0,1}$ , large thick dashes for  $F_{1,1}$ , continuous thin line for  $F_{2,1}$  and continuous thick line for  $F_{3,1}$ .

Note that both  $K_s$  and  $K_w$  are smaller than unity. This implies that the corrections from both ends remain smaller than unity at  $g = 1$  where the leading order weak and the strong coupling curves meet. Given this it is not unreasonable to expect that string perturbation theory may be able to give a fairly good result for the function  $F(g)$  over the entire range of  $g$ .

We shall now follow the procedure outlined in §2 to find the interpolating functions whose Taylor series expansion around  $g = 0$  and/or  $g = \infty$  agree with those of  $F^W(g)$  and/or  $F^S(g)$ . We find

$$F_{0,1}(g) = g^{1/4} (1 + 4 K_s g + g^2)^{1/4}, \quad (5.2)$$

$$F_{1,1}(g) = g^{1/4} (1 + 6 K_s g^2 + g^3)^{1/6}, \quad (5.3)$$

$$F_{2,0}(g) = g^{1/4} (1 + 6 K_w g^2 + g^3)^{1/6}, \quad (5.4)$$

$$F_{2,1}(g) = g^{1/4} (1 + 8 K_w g^2 + 8 K_s g^3 + g^4)^{1/8}, \quad (5.5)$$

$$F_{3,0}(g) = g^{1/4} (1 + 8 K_w g^2 + g^4)^{1/8}, \quad (5.6)$$

and

$$F_{3,1}(g) = g^{1/4} (1 + 10 K_w g^2 + 10 K_s g^4 + g^5)^{1/10}. \quad (5.7)$$

Fig. 2 shows the weak and strong coupling results  $F_2^W$  and  $F_1^S$  including the first subleading corrections and the interpolating function  $F_{3,1}(g)$ . Fig.3 shows the ratios  $F_2^W(g)/F_{3,1}(g)$  and  $F_1^S(g)/F_{3,1}(g)$  as a function of  $\tan^{-1} g$ . We see that these ratios remain close to unity during



$(m, n)$	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
$F''_{m,n}(1)/F_{m,n}(1)$	-0.125	-0.103	0	-0.055	-0.017	-0.066	0.010	-0.006

Table 1: Table showing the value of  $F''_{m,n}(1)/F_{m,n}(1)$  for different  $(m, n)$ .

most of the range of  $g$  except at very large and small values of  $g$  where  $F_2^W(g)$  and  $F_1^S(g)$  respectively are clearly bad approximations to the actual function  $F(g)$ . More specifically, if we denote by  $g_c$  the point where  $F_2^W(g)$  and  $F_1^S(g)$  meet, then  $F_2^W(g)$  agrees with  $F_{3,1}(g)$  below  $g_c$  within 8% and  $F_1^S(g)$  agrees with  $F_{3,1}(g)$  above  $g_c$  within 8%.

To estimate how close we are to the best possible approximation to the full function  $F(g)$ , we can analyze how the interpolating functions at different orders differ from each other. To test this we have plotted in Fig. 4 the ratio of  $F_{m,n}(g)/F_{3,1}(g)$  for  $0 \leq m \leq 3$ ,  $0 \leq n \leq 1$  as a function of  $\tan^{-1} g$ . As we can see, most of the ratios remain within 10% of unity throughout the whole range of  $g$ , indicating that  $F_{3,1}(g)$  may be within 10% of the best possible approximation. In particular we see that  $F_{3,1}$  and  $F_{2,1}$  lie within 5% of each other over the entire range of  $g$ .

Another crude test for determining how good the interpolation formula is its smoothness. This in turn can be determined by computing  $F''_{m,n}(g)/F_{m,n}(g)$  around the matching region  $g \sim 1$ . In Table 1 we have shown the value of  $F''_{m,n}(1)/F_{m,n}(1)$  for various interpolating functions. As we can see, this ratio is largest for the functions  $F_{0,0}$  and  $F_{0,1}$  – precisely the two functions whose deviation from  $F_{3,1}$  is maximum in Fig. 4. This gives us indirect indication that  $F_{m,n}$  for  $(m, n)$  other than (0,0) and (0,1) are smoother, and hence are likely to be better approximations to the actual result than  $F_{0,0}$  or  $F_{0,1}$ .

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## A Testing the algorithm with test functions

In this appendix we shall test the efficiency of the algorithm outlined in §2 by applying it on some test functions.

- The first function we consider is

$$F(g) = (1 + g + g^2)^{1/2}. \quad (\text{A.1})$$

The successive approximations  $F_{m,n}(g)$  to the function  $F(g)$  are taken to be of the form

$$(1 + a_1g + \cdots a_mg^m + b_ng^{m+1} + \cdots b_1g^{m+n} + g^{m+n+1})^{1/(m+n+1)}, \quad (\text{A.2})$$

where the coefficients  $a_1, \cdots a_m$  are fixed by requiring that the Taylor series expansion of  $F_{m,n}(g)$  around  $g = 0$  matches those of  $F(g)$  up to order  $g^m$ , and the coefficients  $b_1, \cdots b_n$  are fixed by requiring that the Taylor series expansion of  $F_{m,n}(g)$  around  $g = \infty$  matches those of  $F(g)$  up to order  $g^{-n+1}$ . For  $m + n + 1$  even,  $F_{m,n}$  and  $F$  are exactly equal since we get  $1 + a_1g + \cdots a_mg^m + b_ng^{m+1} + \cdots b_1g^{m+n} + g^{m+n+1} = (1 + g + g^2)^{(m+n+1)/2}$ . For odd  $m + n + 1$  we get

$$\begin{aligned} F_{0,0}(g) &= 1 + g, & F_{1,1}(g) &= (1 + 3g/2 + 3g^2/2 + g^3)^{1/3}, \\ F_{2,2}(g) &= (1 + 5g/2 + 35g^2/8 + 35g^3/8 + 5g^4/2 + g^5)^{1/5}, \\ F_{3,3}(g) &= (1 + 7g/2 + 63g^2/8 + 175g^3/16 + 175g^4/16 + 63g^5/8 + 7g^6/2 + g^7)^{1/7}, \\ F_{4,4}(g) &= (1 + 9g/2 + 99g^2/8 + 357g^3/16 + 3843g^4/128 + 3843g^5/128 + 357g^6/16 \\ &\quad + 99g^7/8 + 9g^8/2 + g^9)^{1/9}, \end{aligned} \quad (\text{A.3})$$

etc. Using these we find that over the entire range of  $g$

$$\begin{aligned} \left| \frac{F_{0,0}}{F} - 1 \right| &< .155, & \left| \frac{F_{1,1}}{F} - 1 \right| &< .013, & \left| \frac{F_{2,2}}{F} - 1 \right| &< .0021, \\ \left| \frac{F_{3,3}}{F} - 1 \right| &< .00043, & \left| \frac{F_{4,4}}{F} - 1 \right| &< .0001, \end{aligned} \quad (\text{A.4})$$

etc. Thus the error decreases for large  $m, n$ .

- The second test function we consider is

$$F(g) = 2g^{1/4}(1 - e^{-1/g})^{1/2}(1 + e^{-g})^{-1} + g^{3/4}e^{-1/g}. \quad (\text{A.5})$$

Note that there are non-perturbative corrections at both ends and hence we do not expect to approach the exact result even by going to arbitrary high order. The approximation  $F_{m,n}(g)$  is again taken to be of the form

$$g^{1/4}(1 + a_1g + \cdots a_mg^m + b_ng^{m+1} + \cdots b_1g^{m+n} + g^{m+n+1})^{1/2(m+n+1)}. \quad (\text{A.6})$$

The successive approximations are given by

$$\begin{aligned}
F_{0,0}(g) &= g^{1/4}(1+g)^{1/2} \\
F_{0,1}(g) &= g^{1/4}(1+4g+g^2)^{1/4} \\
F_{1,0}(g) &= g^{1/4}(1+2g+g^2)^{1/4} \\
F_{1,1}(g) &= g^{1/4}(1+3g+6g^2+g^3)^{1/6} \\
F_{2,2}(g) &= g^{1/4}(1+5g+45g^2/4+45g^3+10g^4+g^5)^{1/10} \\
F_{3,3}(g) &= g^{1/4}(1+7g+91g^2/4+539g^3/12+2905g^4/8+91g^5+14g^6+g^7)^{1/14} \\
F_{4,4}(g) &= g^{1/4}(1+9g+153g^2/4+405g^3/4+1479g^4/8 \\
&\quad +97323g^5/32+6519g^6/8+153g^7+18g^8+g^9)^{1/18}, \tag{A.7}
\end{aligned}$$

etc. From this we find that over the entire range of  $g$

$$\begin{aligned}
\left| \frac{F_{0,0}}{F} - 1 \right| &< .119, & \left| \frac{F_{1,1}}{F} - 1 \right| &< .064, & \left| \frac{F_{2,1}}{F} - 1 \right| &< .077, & \left| \frac{F_{2,2}}{F} - 1 \right| &< .037, \\
\left| \frac{F_{3,3}}{F} - 1 \right| &< .043, & \left| \frac{F_{4,4}}{F} - 1 \right| &< .057
\end{aligned} \tag{A.8}$$

etc. This shows that while the  $F_{m,n}$ 's can take us quite close to the actual result, we do not approach arbitrarily close to the exact result by going to higher orders. The best result is obtained for  $F_{2,2}(g)$  which comes within 4% of  $F(g)$  over the entire range of  $g$ .

• So far the functions we have analyzed have all the  $a_i$ 's and  $b_i$ 's positive. We shall now give the example of a function that has some of these coefficients negative but small enough so that the approximation scheme used here still takes us sufficiently close to the exact function. We consider

$$F(g) = (1 - g/5 + g^2)^{1/2}. \tag{A.9}$$

The  $F_{m,n}$ 's are taken to be

$$(1 + a_1g + \cdots a_mg^m + b_ng^{m+1} + \cdots b_1g^{m+n} + g^{m+n+1})^{1/(m+n+1)}. \tag{A.10}$$

In this case again for  $m+n+1$  even,  $F_{m,n}$  and  $F$  are exactly equal. For odd  $m+n+1$  we get

$$\begin{aligned}
F_{0,0}(g) &= 1 + g, & F_{1,1}(g) &= (1 - 0.3g - 0.3g^2 + g^3)^{1/3}, \\
F_{2,2}(g) &= (1 - 0.5g + 2.575g^2 + 2.575g^3 - 0.5g^4 + g^5)^{1/5}, \\
F_{3,3}(g) &= (1 - 0.7g + 3.675g^2 - 1.7675g^3 - 1.7675g^4 + 3.675g^5 - 0.7g^6 + g^7)^{1/7}, \\
F_{4,4}(g) &= (1 - 0.9g + 4.815g^2 - 3.2025g^3 + 8.66644g^4 + 8.66644g^5 - 3.2025g^6 \\
&\quad + 4.815g^7 - 0.9g^8 + g^9)^{1/9}, \tag{A.11}
\end{aligned}$$

etc. Using these we find that over the entire range of  $g$

$$\begin{aligned} \left| \frac{F_{0,0}}{F} - 1 \right| &< .5, & \left| \frac{F_{1,1}}{F} - 1 \right| &< .17, & \left| \frac{F_{2,2}}{F} - 1 \right| &< .08, \\ \left| \frac{F_{3,3}}{F} - 1 \right| &< .08, & \left| \frac{F_{4,4}}{F} - 1 \right| &< .05, \end{aligned} \quad (\text{A.12})$$

etc. However the error does not reduce uniformly; instead it fluctuates around the 5% mark as we go to higher order. Since the sign of the error fluctuates, we can considerably reduce the size of the error by averaging the result for different values of  $m, n$ .

• Finally we consider the function

$$F(g) = \int_{-\infty}^{\infty} dx e^{-x^2/2 - g^2 x^4}. \quad (\text{A.13})$$

The expansion around  $g = 0$  is known to be asymptotic. Since the function goes as  $\sqrt{2\pi}$  as  $g \rightarrow 0$  and as  $(4g)^{-1/2}\Gamma(1/4)$  as  $g \rightarrow \infty$  we take

$$\begin{aligned} F_{m,n}(g) = \sqrt{2\pi} \Big\{ &1 + a_1 g + \cdots a_m g^m + b_n g^{m+1} + \cdots b_1 g^{m+n} \\ &+ (8\pi)^{(m+n+1)} \Gamma(1/4)^{-2(m+n+1)} g^{m+n+1} \Big\}^{-1/\{2(m+n+1)\}}, \end{aligned} \quad (\text{A.14})$$

and adjust the coefficients  $a_k$  and  $b_k$  so as to reproduce the series expansion around  $g = 0$  and  $g = \infty$ . Note that in this case the weak coupling expansion is in powers of  $g^2$  whereas the strong coupling expansion is in powers of  $1/g$ , exactly as in our case. We find that  $F_{m,n}(g)$  approaches quite close to  $F(g)$  for large enough  $(m, n)$ . For example

$$F_{4,4}(g) = \sqrt{2\pi} \left\{ 1 + 54 g^2 + 594 g^4 + 780.788 g^5 + 1294.34 g^6 + 1475.35 g^7 + 1038.59 g^8 + 341.428 g^9 \right\}^{-1/18} \quad (\text{A.15})$$

agrees with  $F(g)$  within 0.4% over the entire range of  $g$ .

## B Determination of the normalization constant $\mathcal{N}$

In this appendix we shall determine the normalization constant  $\mathcal{N}$  appearing in (4.13) by comparing this with the corresponding expression for  $\delta M^2$  in the field theory limit. Let  $\chi_s$  denote a massive scalar in the spinor representation of the  $\text{SO}(32)$  gauge group. Then the action of the low energy effective field theory describing the  $\text{SO}(32)$  gauge fields and their coupling to the scalars  $\chi_s$  is given by

$$\int d^{10}x \left[ -\frac{1}{16(g_H)^2} \text{Tr}_V(F_{\mu\nu} F^{\mu\nu}) - D_\mu \chi_s^* D^\mu \chi_s - M^2 \chi_s^* \chi_s \right]. \quad (\text{B.1})$$

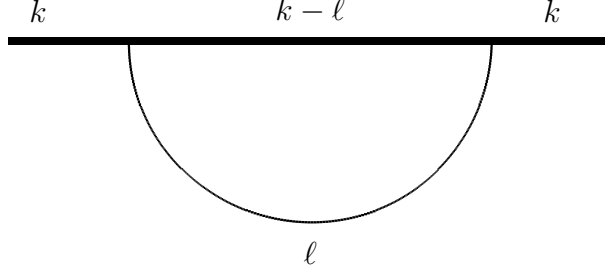


Figure 5: A Feynman diagram of low energy effective field theory contributing to mass renormalization. The thick line represents the propagator of the massive scalar in the spinor representation of  $SO(32)$  and the thin line represents a gauge boson propagator.

Note that the gauge field action has been normalized in accordance with (2.1) by setting  $G_{\mu\nu}^H = \eta_{\mu\nu}$  and the normalization of the kinetic term for  $\chi_s$  is standard. We shall compute the mass renormalization of  $\chi_s$  due to the one loop diagram of Fig. 5. More specifically we shall compute the contribution from the region of loop momentum integration where the momentum  $\ell$  carried by the vector fields is small. In this limit we essentially compute the correction to the mass due to the Coulomb field of the  $\chi$  particle and the result is independent of the spin of  $\chi$ . Indeed it is easy to verify that our results remain unchanged if instead of  $\chi$  we use another field with a different spin *e.g.* a Dirac field. This is important since the  $SO(32)$  spinor states for which we need to compute the mass renormalization in string theory are not scalars but transform in the **84+44** representation of the little group  $SO(9)$ .

In order to compute the contribution from the graph shown in Fig 5 we first need to choose a convention for the  $SO(32)$  generators  $T^a$ . In the vector representation we choose them to be  $32 \times 32$  matrices whose  $mn$  element is  $i$ ,  $nm$  element is  $-i$  and whose other elements are zero. Collection of all such matrices for  $1 \leq m < n \leq 32$  form the basis of  $SO(32)$  generators. With this convention we have  $Tr_V(T^a T^b) = 2\delta_{ab}$  and hence the gauge kinetic term is normalized as  $-\frac{1}{8}F_{\mu\nu}^a F^{a\mu\nu}$ . As a result the gauge field propagator of momentum  $\ell$  in the Feynman gauge will be of the form  $-2i\delta_{ab}g_{\mu\nu}/(\ell^2 - i\epsilon)$ . The coupling of the gauge field to  $\chi$  in the Lgrangian density is of the form  $iA^{a\mu}(T^a)_{ss'}(\chi_s^* \partial_\mu \chi_{s'} - \partial_\mu \chi_s^* \chi_{s'})$  from which we can read out the vertex. This gives the contribution to  $\delta M^2$  to be

$$-i\delta M^2 = 2(T^a T^a)_{spinor} \int \frac{d^{10}\ell}{(2\pi)^{10}} \frac{1}{\ell^2 - i\epsilon} \frac{1}{(k - \ell)^2 + M^2 - i\epsilon} (2k_\mu - \ell_\mu)(2k^\mu - \ell^\mu) \quad (B.2)$$

where the momentum integration is still written in the Lorentzian space, and  $(T^a T^a)_{spinor}$

denotes the eigenvalue of  $T^a T^a$  on the spinor representation. With the normalization of  $T^a$  we have chosen, each  $T^a$  has eigenvalue  $\pm 1/2$  on the spinor representation and hence, after summing over the 496 generators of  $\text{SO}(32)$ ,  $T^a T^a$  will have eigenvalue 124 on the spinor representation.

We shall analyze the contribution to (B.2) from the region of integration where  $\ell$  is small so that the low energy effective field theory description makes sense. In this case we can replace the vertex factor  $(2k_\mu - \ell_\mu)(2k^\mu - \ell^\mu)$  by  $4k_\mu k^\mu = -4(k_0)^2$  and take it out of the integral. Making the Wick rotation  $\ell^0 \rightarrow i\ell^0$  and denoting by  $\ell_E$  the Euclidean momentum we get

$$\delta M_{gauge}^2 = 8(k_0)^2 \times 124 \times \int \frac{d^{10}\ell_E}{(2\pi)^{10}} \frac{1}{\ell_E^2} \frac{1}{(\ell_E - k)^2 + M^2}. \quad (\text{B.3})$$

In order to make connection with the string theory result (4.13) we now need to express the propagators in the Schwinger proper time formalism. We write

$$\delta M_{gauge}^2 = 992(k_0)^2 \pi^2 \int \frac{d^{10}\ell_E}{(2\pi)^{10}} \int_0^\infty dy_2 \int_0^\infty dz_2 e^{-\pi y_2 \ell_E^2 - \pi z_2 ((\ell_E - k)^2 + M^2)}. \quad (\text{B.4})$$

Changing variable to  $\tau_2 = y_2 + z_2$  and  $z_2$  we get

$$\begin{aligned} \delta M_{gauge}^2 &= 992(k_0)^2 \pi^2 \int \frac{d^{10}\ell_E}{(2\pi)^{10}} \int_0^\infty d\tau_2 \int_0^{\tau_2} dz_2 e^{-\pi(\tau_2 - z_2)\ell_E^2 - \pi z_2 ((\ell_E - k)^2 + M^2)} \\ &= 992(k_0)^2 \pi^2 \int \frac{d^{10}\ell_E}{(2\pi)^{10}} \int_0^\infty d\tau_2 \int_0^{\tau_2} dz_2 e^{-\pi\tau_2(\ell_E - z_2 k/\tau_2)^2 + \pi z_2^2 k^2/\tau_2 - \pi z_2(k^2 + M^2)} \\ &= 992(k_0)^2 \pi^2 (2\pi)^{-10} \int_0^\infty d\tau_2 \int_0^{\tau_2} dz_2 (\tau_2)^{-5} e^{-4\pi z_2^2/\tau_2}, \end{aligned} \quad (\text{B.5})$$

where in the last step we have carried out the integration over  $\ell_E$  and have also used the on-shell condition  $k^2 = -M^2 = -4$  in the exponent.

We shall now try to reproduce the same integral from the string theory result (4.13). We shall focus on the region of integration where both  $z_2$  and  $\tau_2$  are large but  $z_2$  remains small compared to  $\tau_2$ . Since the integrand in (4.13) is invariant under  $z \rightarrow \tau - z$ , there will be an identical contribution from the region where  $(\tau_2 - z_2)$  is small compared to  $\tau_2$ , and the effect of this will be to double the contribution from the  $z_2 \ll \tau_2$  region.<sup>6</sup> Now since in computing the field theory contribution we examined only the graphs with bosonic intermediate states, we must do the same in string theory. This corresponds to restricting the sum over spin structure

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<sup>6</sup>Physically  $z \rightarrow \tau - z$  transformation exchanges the role of the two arms of the torus between the points 0 and  $z$ .

$\nu$  to 00 and 01 sectors only. We now use the following approximations to the various factors in (4.13) for large  $\tau_2, z_2$ :<sup>7</sup>

$$\begin{aligned}
\vartheta_{00}(z)^4 &\simeq 1 + 4e^{\pi i\tau - 2\pi iz} + 4e^{\pi i\tau + 2\pi iz} + \dots \\
\vartheta_{01}(z)^4 &\simeq 1 - 4e^{\pi i\tau - 2\pi iz} - 4e^{\pi i\tau + 2\pi iz} + \dots \\
\vartheta_{11}(z)^{-1} &\simeq -ie^{-\pi i\tau/4}e^{\pi iz}(1 + \dots) \\
\eta(\tau)^{-1} &\simeq e^{-\pi i\tau/12}(1 + \dots) \\
\overline{\vartheta_{00}(z/2)}^{16} &\simeq 1 + 16e^{-\pi i\bar{\tau} + \pi i\bar{z}} + 16e^{-\pi i\bar{\tau} - \pi i\bar{z}} + 120e^{-2\pi i\bar{\tau} + 2\pi i\bar{z}} + \dots \\
\overline{\vartheta_{01}(z/2)}^{16} &\simeq 1 - 16e^{-\pi i\bar{\tau} + \pi i\bar{z}} - 16e^{-\pi i\bar{\tau} - \pi i\bar{z}} + 120e^{-2\pi i\bar{\tau} + 2\pi i\bar{z}} + \dots \\
\overline{\vartheta_{10}(z/2)}^{16} &\simeq 0 \\
\overline{\vartheta_{11}(z/2)}^{16} &\simeq 0 \\
(\overline{\vartheta_{11}(z)})^{-1} &\simeq ie^{\pi i\bar{\tau}/4}e^{-\pi i\bar{z}}(1 - e^{-2\pi i\bar{z}} - e^{-2\pi i(\bar{\tau} - \bar{z})} + \dots)^{-1} \\
&\simeq e^{\pi i\bar{\tau}/4}e^{-\pi i\bar{z}}(1 + e^{-2\pi i\bar{z}} + e^{-4\pi i\bar{z}} + \dots + e^{2\pi i(\bar{\tau} - \bar{z})} + \dots), \\
(\overline{\eta(\tau)})^{-1} &\simeq e^{\pi i\bar{\tau}/12}(1 + e^{-2\pi i\bar{\tau}} + \dots). \tag{B.6}
\end{aligned}$$

Substituting these into (4.13), and taking into account the extra factor of 2 due to  $z \rightarrow \tau - z$  symmetry, we get the string theory result for  $\delta M^2$  from the region where  $z_2$  and  $\tau_2$  are large:

$$\begin{aligned}
\delta M^2 &\simeq 8 \times \mathcal{N}(k_0)^2 \int_{\tau_2 \gg 1} d^2\tau \int_{1 < z_2 < \tau_2} d^2z e^{2\pi i(\bar{\tau} - \bar{z})} \\
&\left\{ (1 + 4e^{-2\pi i\bar{\tau}} + \dots)(1 + 4e^{-2\pi i\bar{z}} + 4e^{-2\pi i\bar{\tau} + 2\pi i\bar{z}} + \dots)((1 + 120e^{-2\pi i\bar{\tau} + 2\pi i\bar{z}} + \dots)) \right\} \\
&\{1 + \dots\} \{ (1 + 2e^{-2\pi i\bar{z}} + 2e^{-2\pi i(\bar{\tau} - \bar{z})} + \dots)(1 + 14e^{-2\pi i\bar{\tau}} + \dots)e^{-4\pi z_2^2/\tau_2} (\tau_2)^{-5} \}. \tag{B.7}
\end{aligned}$$

This expression has been organized as follows. After factoring out the leading terms inside each of the curly brackets in (4.13), we have written inside the three curly brackets the subleading terms from the terms inside the three curly brackets in (4.13). In particular since the leading holomorphic term has no factors of  $e^{-2\pi i\tau}$  or  $e^{-2\pi iz}$  we have dropped all the subleading pieces containing factors of  $e^{2\pi i\tau}$  or  $e^{2\pi iz}$ .

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<sup>7</sup>Note that the leading term in the integrand has a stronger singularity in  $\bar{\tau}$  compared to  $\tau$  and hence in the expansion in  $e^{-2\pi i\bar{\tau}}$  and  $e^{-2\pi i\bar{z}}$  we must keep more terms than in their holomorphic counterparts. The general rule is that while expanding the holomorphic terms we only keep the leading terms except in those inside the sum over the spin structure  $\nu$  where we have to keep the first subleading terms. On the other hand for the anti-holomorphic terms we need to keep up to subleading terms of order  $e^{-2\pi i\bar{\tau}}$ ,

Now for large  $\tau_2$  the integrals over  $\tau_1$  and  $z_1$  run from  $-1/2$  to  $1/2$  without any restriction. Carrying out these integrals projects us into those terms which do not have any factors of  $e^{-2\pi i\bar{\tau}}$  or  $e^{-2\pi i\bar{z}}$ . Thus we need to identify such terms in (B.7). Furthermore since we are only interested in picking up the contributions due to gauge boson exchange we need to consider only the subleading terms coming from the expansion of the terms inside the first curly bracket since this is what arose from the 32 left-moving fermions. The  $e^{-2\pi i(\bar{\tau}-\bar{z})}$  term in this expansion, which cancels the overall multiplicative factor of  $e^{2\pi i(\bar{\tau}-\bar{z})}$ , has coefficient  $(120+4) = 124$ . Thus the gauge boson exchange contribution to (B.7) is given by

$$8 \times 124 \mathcal{N} (k_0)^2 \int d^2\tau \int d^2z e^{-4\pi z_2^2/\tau_2} (\tau_2)^{-5}. \quad (\text{B.8})$$

Comparing this with the field theory result (B.5) we now get

$$\mathcal{N} = \frac{992}{8 \times 124} \pi^2 (2\pi)^{-10} = 2^{-10} \pi^{-8}. \quad (\text{B.9})$$

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